Exact Solution Techniques for Large Location Problems

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Abstract: Most problems, which arise from planning and managing distribution systems, are known to be NP-hard. Solving the problems of considerable size to optimality, a huge increase of computation intricacy must be awaited in a common case. It is the reason, for which this problems are solved using various heuristics to get at least suboptimal solution of the problems from practice. There is one problem, among the distribution problems, which is NP-hard also, but computational effort of its exact solution does not seem to be as huge as in the other cases. The problem is known as uncapacitated location problem. In the problem, facilities of unrestricted size are placed among m possible places with the objective of minimising the total cost for satisfying given demands of n customers. Costs include a fixed charge for facility location at place i and cost of satisfying j-th customer's demand from place i. In contradiction to the other problems, the branch and bound method employed to the location problem can bring exact solution in sensible time even if the size of the problem is considerably large. We shall present our computational results with our implementation of the exact method applied to problems with more than 450 possible places of location. The influence of the various ways of lower bound technique on the computational time and number of fathomed nodes will be reported as well.

Keywords: Uncapacitated location problem, branch and bound, lower bound.

1. Introduction

When planning large distribution system, an uncapacitated location problem is often met. The location problem consists in placing some facilities of unrestricted capacity among sites from a given set I with the objective of minimising the total cost for satisfying fixed demands specified at locations from given set J. Costs consist of a fixed charge for building each facility and amount for satisfying j's demand from facility i. The above mentioned problem arises e.g. when network of stores is created to serve an estimated set of future customers under assumption that unit cost of goods transport from factory to store is much cheaper than unit cost of transport between store and customer. Another application emerges, when a decision is made which station will be established as train-forming station and which station will be served only in a railway network. In both examples a large set J of served nodes (customers or served stations) and finite set I of possible locations (stores or train-forming stations) are given. We assume that for each possible location $i \in I$ a fixed charge f_i is given and that for each pair (i,j), where $i \in I$ and $j \in J$ the cost c_{ij} of serving demand of customer j from location i is known. Next instants of the problem can be found in [2], [3], [7].

To solve the problem exactly, several algorithms were developed, but only branch and bound based algorithms proved to be able to solve the problems of considerable large size. In this contribution, we focus at lower bound computation, because the tightness of the lower bound influences to a large extent the time of computation. We compare lower bound due Erlenkotter [1], Kubat [7] and we study LP-relaxation of the week formulation of the location problem too.

$$\text{subject to} \quad \sum_{i \in V \cup N} x_{ij} = 1 \qquad \text{for } j \in J \qquad (7)$$

$$y_i - x_{ij} \geq 0 \qquad \text{for } i \in N \text{ and } j \in J \qquad (8)$$

$$x_{ij} \geq 0 \qquad \text{for } i \in V \cup N \text{ and } j \in J \qquad (9)$$

$$y_i \geq 0 \qquad \text{for } i \in N \qquad (10)$$

$$y_i \leq 1 \qquad \text{for } i \in N \qquad (11)$$

Considering only positive constants f_i, constraints *(11)* can be skipped without loss of generality. A dual problem to model *(6)-(10)* is

$$\text{maximise} \quad g(v,w) = \sum_{j \in J} v_j \qquad (12)$$

$$\text{subject to} \quad v_j - w_{ij} \leq c_{ij} \qquad \text{for } i \in N \text{ and } j \in J \qquad (13)$$

$$v_j \leq c_{ij} \qquad \text{for } i \in V \text{ and } j \in J \qquad (14)$$

$$\sum_{j \in J} w_{ij} \leq f_i \qquad \text{pro } i \in N \qquad (15)$$

$$w_{ij} \geq 0 \qquad \text{pro } i \in N, j \in J \qquad (16)$$

Constraints *(13)* and */16)* imply $max\{v_j - c_{ij}, 0\} \leq w_{ij}$ and it is obvious that it can be required equality in the constraint for optimal solution of *(12)-(16)*. Then problem *(12)-(16)* takes form:

$$\text{maximise} \quad g(v) = \sum_{j \in J} v_j \qquad (17)$$

$$\text{subject to} \quad \sum_{j \in J} max\{v_j - c_{ij}, 0\} \leq f_i \qquad \text{for } i \in N \qquad (18)$$

$$v_j \leq min\{c_{ij} : i \in V\} \qquad \text{for } j \in J \qquad (19)$$

Computing lower bound E_{LB}, the dual ascent algorithm [1] is used to get good feasible solution of *(17)-(19)*. The lower bound is further improved by dual adjustment algorithm and the process is repeated until improvement of *(17)* is obtained.

2.3 Kubat's Lower Bound

Kubat's lower bound determination issues from properties of a location problem solution, which belongs to a given search tree node given by the sets V, Z, N. The solution is determined by set P of locations in which facilities are to be placed. Constraints $P \subseteq V \cup N$ and $V \subseteq P$ must hold for P. The objective function value of the solution is given by

$$f(P) = \sum_{i \in P} f_i + \sum_{j \in J} min\{c_{ij} : i \in P\}$$

Let us denote J_V set of customers $j \in J$ for which $min\{c_{ij} : i \in V\} = min\{c_{ij} : i \in V \cup N\}$ holds. If we denote $J_N = J - J_V$ and $/J_N/$ number of elements of set J_N, then we get

$$f(P) = \sum_{i \in V} f_i + \sum_{j \in J_V} min\{c_{ij} : i \in P\} + \sum_{k \in P-V} f_k + \sum_{j \in J_N} min\{c_{ij} : i \in P\} = \sum_{i \in V} f_i +$$
$$\sum_{j \in J_V} min\{c_{ij} : i \in V\} + \sum_{j \in J_N} \sum_{k \in P-V} f_k / /J_N/ + \sum_{j \in J_N} min\{c_{ij} : i \in P\} = \sum_{i \in V} f_i + \sum_{j \in J_V} min\{c_{ij} : i \in V\} +$$
$$\sum_{j \in J_N} min\{ \sum_{k \in P-V} f_k / /J_N/ + c_{ij} : i \in V \cup (P-V)\} = \sum_{i \in V} f_i + \sum_{j \in J_V} min\{c_{ij} : i \in V\} +$$

2. Mathematical Model and Exact Methods

Let introduce 0-1 variable y_i for each possible facility location $i \in I$ to describe decision of placing $(y_i=1)$ a facility at the location or opposite decision $(y_i=0)$. Let denote x_{ij} the fraction of j's demand supplied from the facility i. Then employing the above mentioned constants f_i and c_{ij} we can form the following model:

minimise $$f(x,y) = \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \qquad (1)$$

subject to $$\sum_{i \in I} x_{ij} = 1 \qquad \text{for } j \in J \qquad (2)$$

$$x_{ij} \leq y_i \qquad \text{for } i \in I \text{ and } j \in J \qquad (3).$$

$$x_{ij} \geq 0 \qquad \text{for } i \in I \text{ and } j \in J \qquad (4)$$

$$y_i \in \{0,1\} \qquad \text{for } i \in I \qquad (5)$$

In the model, constraints *(1)* ensure that each customer demand is served. Constraints *(2)* force to place a facility at location i whenever any part of demand of any customer is served from the location.

2.1 Branch and Bound Method

Having used the „Depth first scheme" for tree search, a procedure of branch and bound method used for location problem has the following form:

A node of the search tree is determined by three disjoint subsets which cover set I of possible facility locations, let us denote them V, Z and N. The subsets contain locations, where decision of facility placing was made (V), where facility placing was forbidden (Z) and where no decision has been made (N). The node represents a set of feasible solutions of the location problem, which values of y variables satisfy constraints given by sets V, Z. When investigating the node, a lower bound on objective function value of all feasible solutions of the node is determined. If the lower bound is higher than objective function value of the current best feasible solution, then the node is said to be fathomed and backtracking to previous node (father) is made. In the opposite case the node is investigated using branching procedure. The procedure consist in selecting a location from set N and two new nodes (sons) are formed making decision of facility placing at the location. The first of the sons has forbidden placing at the location and the second has ordered placing. The search continues by first son investigation and when the search comes back to the node, then it continues by second son investigation. When the second son is fathomed and the search comes back to the node second times, then the node is declared fathomed and backtracking to previous node is made as well. During lower bound computation a feasible solution of the location problem is usually produced. This feasible solution is compared with the current best solution and the current solution is updated. It is obvious that number of investigated nodes and the time of whole search depend on lower bound quality. That is why, we focus at various approaches to determination of location problem lower bound. To specify an individual approaches the set notation V, Z, N will be used.

2.2 Erlenkotter's Lower Bound

Erlenkotter's lower bound [1] is obtained as lower bound estimation of optimal objective function value of LP-relaxation of the model *(1)-(5)* with respect to sets V, Z, N. In this approach, objective value of a dual feasible solution serves as the lower estimation. The primal model of LP-relaxation is considered in the form:

minimise $$f(x,y) = \sum_{i \in V} f_i + \sum_{i \in N} f_i y_i + \sum_{i \in V \cup N} \sum_{j \in J} c_{ij} x_{ij} \qquad (6)$$

$$\sum_{j\in J_N} min\{ min\{ \sum_{k\in P-V} f_k / |J_N| + c_{ij} : i\in V\},\ min\{ \sum_{k\in P-V} f_k / |J_N| + c_{ij} : i\in P-V\}\} \geq$$

$$\sum_{i\in V} f_i + \sum_{j\in J_V} min\{c_{ij} : i\in V\} + \sum_{j\in J_N} min\{ min\{ c_{ij} : i\in V\},\ min\{ f_i / |J_N| + c_{ij} : i\in N\}\} = K_{LB} \quad (20)$$

The right-hand-side of the last inequality is so called Kubat's lower bound for a search tree node characterised by the sets *V, Z, N.*

2.4 LP-Relaxation Lower Bound

Another possibility to get lower bound on feasible solution values of search tree node *(V, Z, N)* is to solve to optimality LP-relaxation *(6)-(10)*, where constraints *(8)* are replaced by their weaker form *(21)*.

$$\sum_{j\in J} c_{ij}\, x_{ij} \leq |J| y_i \qquad for\ i \in N \qquad (21)$$

Using *(21)* and realising that constraints *(21)* will be satisfied as equalities for optimal solution of *(6), (7), (21), (9), (10)*, we can use the equality as substitutional formula and we get model

$$minimise \qquad f(x,y) = \sum_{i\in V} f_i + \sum_{j\in J} (\sum_{i\in V} c_{ij}\, x_{ij} + \sum_{i\in N} (f_i / |J| + c_{ij}) x_{ij}) \qquad (22)$$

subject to *(7), (9)*

Optimal solution value of *(22),(7),(9)* is

$$\sum_{i\in V} f_i + \sum_{j\in J} min\{ min\{ c_{ij} : i\in V\},\ min\{ f_i / |J| + c_{ij} : i\in N\}\} = R_{LB} \qquad (23)$$

and this value is LP-relaxation lower bound R_{LB}. Comparing *(23)* with *(20)*, we can state that $R_{LB} \leq K_{LB}$. Thus, the only question remains if and how much is E_{LB} better than K_{LB}.

3. Computational Study

To decide the question if E_{LB} can be improved by K_{LB} in some instants, the following experiments were carried out. A procedure for K_{LB} computation was coded and used in branch and bound program together with E_{LB} computation, whenever a node is investigated. Both lower bounds were compared and improvement or deterioration was evaluated. The improvement was measured by value of $(K_{LB} - E_{LB})/Gap$, where *Gap* is difference between current upper bound (the value of the current best solution) and E_{LB}. The deterioration was measured by value of $(E_{LB} - K_{LB})/Gap$. For individual instances number of fathomed nodes (Nodes), number of improvements (Nimp), or deteriorations (Ndet) were recorded together with average improvement (Aimp) or deteriorations (Adet) and time of computations in seconds (Time). The experiments were carried out on a model of Slovak railway network with a set of 457 significant stations [5], [6]. A given percentage of the stations was taken as possible facility locations (train-forming stations) to form set *I*. The percentages were 10%, 20%, 30%, 40%, 50%, 60% and associated notation of the class of instants were N02?, N04?, N06?, N08?, N10? and N12? respectively. Costs c_{ij} were computed as product of randomly generated demand (from 5 to 15 carriages) and unit cost, where the unit cost was determined according to formula $60+ 20\, d_{ij}$, where d_{ij} was distance between *i* and *j* A constant fixed charge was used for each instant of a class. The charge was chosen from set 2.5 , 5, 10, 20,..,70 (10^3) . This way, nine instances were obtained for each class with given set of facility locations. The experiments were made on personal computer Pentium 75 Mhz and showed that in no instant K_{LB} was better than E_{LB}. Average results and standard deviations of nine instances for each class determined by set *I* are reported in table 1.

Tab. 1

Network	AvgNodes	StdNodes	AvgAdet	StdAdet	AvgTime	StdTime
N02?	1.1	1.3	159.0	135.4	0.8	0.6
N04?	11.1	11.6	1700.3	3007.1	10.8	11.1
N06?	62.2	92.5	498.6	268.2	76.4	111.9
N08?	60.0	61.0	693.9	910.0	106.3	107.5
N10?	300.1	395.8	633.7	295.7	693.6	967.4
N12?	370.9	565.1	743.9	1055.7	1050.0	1604.4

4. Conclusion

The computational study showed that Erlenkotter's lower bound even if obtained as a lower estimation of the LP-relaxation of model *(1)-(6)* many times overcomes Kubat's lower bound. It follows from comment in 2.4 that the Erlenkotter's lower bound is much better than LP-relaxation lower bound of *(6), (7), (21), (9), (10)* as well.

References

[1] Erlenkotter, D.: A Dual-Based Procedure for Uncapacitated Facility Location. Operations Research, Vol 26, No 6, November-December 1978, pp. 992-1009

[2] Gelders, L.F., Pintelon, L.M., Wassenhove, L.N. van: A location-allocation problem in a large Belgian brewery. European Journal of Operational Research 28, 1987, pp 196-206

[3] Janáček, J.: Optimální umisťování středisek v dopravních sítích. Horizonty dopravy 4/93, č. 4, 1993, s. 12-15

[5] Janáček J., Kovačiková J.: Citlivost doby výpočtu optimálního řešení lokační úlohy na velikosti fixních nákladů. In: sborník príspevkov zo seminára „Kvantitatívne metódy v ekonomike, Fakulta hospodárskej informatiky Ekonomickej univerzity v Bratislave,3.-4.12.1996, s. 193-196

[6] Janáček J., Kovačiková J.: Exaktní řešení lokačních úloh na železniční síti. In: sborník konference ŽEL 96, Žilina 30.-31.5.1996, VŠDS Žilina, s. 87-94

[7] Kubát, J.: Soubor programů pro optimalizaci struktury skladové sítě jedné organizace. Manipulace, skladování a balení č. 5, 1990, SNTL, s. 129-132

[8] Reeves, C., R.: Modern Heuristic Techniques for Combinatorial Problems. Oxford Blackwell Scientific Publications, 1993, 320 s.

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